Generic linear perturbations

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Reference

1. J. N. Mather, *Generic projections*, Ann. of Math., (2) **98** (1973), 226–245.
2. S. Ichiki, *Generic linear perturbations*, arXiv:1607.03220 (preprint).

The celebrated paper [1] by Mather is known as one of the most important papers of singularity theory. By the main theorem in [1], many applications are obtained.
On the other hand, by the main theorem of [2], the main theorem of [1] is drastically improved. Hence, in my talk, the main theorem of [2] and the sketch of the proof are introduced.

Table of Contents

The purpose of this talk is to give the following two:

- the statement of the main theorem (an improvement of the main theorem of "Generic projections")
- the sketch of the proof of the main theorem

Motivation

All mappings and manifolds belong to class C^∞ in this talk.

- $\ell, m, n \in \mathbb{N}$
- N : n-dimensional manifold
- $f : N \to \mathbb{R}^m$: embedding
- $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$: linear mapping

In the paper "Generic projections" by John Mather, the composition $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated from the viewpoint of stability.

Motivation

- $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: set consisting of linear mappings $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$
- There exists the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$.
- By the symbol Σ, we denote a Lebesgue measure zero set of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ $(= (\mathbb{R}^m)^\ell)$.

Motivation

For examples, the following are some of applications obtained in the paper "Generic projections".

- For a given embedding $f : N \to \mathbb{R}^m$,
  ${}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R})$
  s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, $\pi \circ f : N \to \mathbb{R}$ is a Morse function.
- For a given embedding $f : N \to \mathbb{R}^m$ $(n = \dim N \geq 2)$,
  ${}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^2)$ s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^2) - \Sigma$,
  $\pi \circ f : N \to \mathbb{R}^2$ has only fold points and cusp points as the singular points.

and so on.

Motivation

- $F : U \to \mathbb{R}^\ell$: mapping ($U \subset \mathbb{R}^m$: open)
- $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$: linear mapping
- Set

$$F_\pi = F + \pi.$$

In "Generic projections", for a given embedding $f : N \to \mathbb{R}^m$, the composition $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated.
On the other hand, in today's talk (ref., "Generic linear perturbations"), for a given embedding $f : N \to U$, the composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is investigated.

Preliminaries

- N, P : manifold
- $J^r(N,P)$: the space of r-jets of mappings of N into P
- For a given mapping $g : N \to P$, $j^r g : N \to J^r(N,P)$ is defined by $q \mapsto j^r g(q)$.
- $C^\infty(N,P)$: the set of C^∞ mappings of N into P (the topology on $C^\infty(N,P)$: Whitney C^∞ topology)
- $g : N \to P$ is *$\mathcal{A}$-equivalent* to $h : N \to P$ $\overset{def}{\Leftrightarrow}$ $\exists$ diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ s. t. $g = \Psi \circ h \circ \Phi^{-1}$.
- $g : N \to P$: *stable* $\overset{def}{\Leftrightarrow}$ the $\mathcal{A}$-equivalence class of g is open in $C^\infty(N,P)$.

Preliminaries

- $N^{(s)} = \{q = (q_1, \ldots, q_s) \in N^s \mid q_i \neq q_j (1 \leq i < j \leq s)\}$
- ${}_sJ^r(N, P)$ is defined as follows: $\{(j^r g_1(q_1), \ldots, j^r g_s(q_s)) \in J^r(N, P)^s \mid q \in N^{(s)}\}$.
- For $g : N \to P$, the mapping ${}_sj^r g : N^{(s)} \to {}_sJ^r(N, P)$ is defined by $(q_1, \ldots, q_s) \mapsto (j^r g(q_1), \ldots, j^r g(q_s))$.

Preliminaries

- N, P : manifold
- W : submanifold of ${}_sJ^r(N,P)$

Definition 1

- ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ *is* *transverse* *to* W
  $\overset{def}{\Leftrightarrow}$ *for any* $q \in N^{(s)}$, *either one of the following two holds:*
  1. ${}_sj^r g(q) \notin W$
  2. ${}_sj^r g(q) \in W$ *and*
     $d({}_sj^r g)_q(T_q N^{(s)}) + T_{{}_sj^r g(q)} W = T_{{}_sj^r g(q)}\,{}_sJ^r(N,P).$
- $g : N \to P$ *is* *transverse with respect to* W
  $\overset{def}{\Leftrightarrow}$ ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ *is transverse to* W.

Preliminaries

- π : a partition of $\{1, \ldots, s\}$
  $\overset{def}{\Leftrightarrow}$
  1. $^{\exists}A_1, \ldots, A_m \subset \{1, \ldots, s\}$ s. t. $\pi = \{A_1, \ldots, A_m\}$
  2. $\bigcup_{i=1}^{m} A_i = \{1, \ldots, s\}$
  3. $^{\forall}i \; (1 \leq i \leq m)$, $A_i \neq \emptyset$
  4. $^{\forall}i, j \; (i \neq j)$, $A_i \cap A_j = \emptyset$
- Following Mather, we define P^{π} as follows:
  $P^{\pi} = \{(y_1, \ldots, y_s) \in P^s \mid y_i = y_j \iff i$ and j are in the same member of the partition π $\}$

Preliminaries

- Diff N : the group of diffeomorphisms of N.
- There is a natural action of Diff $N \times$ Diff P on ${}_sJ^r(N,P)$ s. t. $(h,H)\cdot {}_sj^rg(q) = {}_sj^r(H\circ g\circ h^{-1})(h(q))$.
- W $(\subset {}_sJ^r(N,P))$: *invariant* $\overset{def}{\Leftrightarrow}$ it is invariant under this action.

Preliminaries

In order to introduce the main theorem of "Generic projections", it is necessary to prepare the notion "modular submanifold " defined by Mather.

- $q = (q_1, \ldots, q_s) \in N^{(s)}$
- $g : N \to P$
- $q' = (g(q_1), \ldots, g(q_s))$

- $J^r(N,P)_{q_i}$: the fiber of $J^r(N,P)$ over q_i $(1 \le i \le s)$
- ${}_sJ^r(N,P)_q$ (resp., ${}_sJ^r(N,P)_{q,q'}$) : the fiber of ${}_sJ^r(N,P)$ over q (resp., over (q,q')).
- $J^r(N)_q = \bigoplus_{i=1}^{s} J^r(N,\mathbb{R})_{q_i}$
- $\mathfrak{m}_q = \bigoplus_{i=1}^{s} \mathfrak{m}_{q_i}$
  $(\mathfrak{m}_{q_i} = \{j^r h_i(q_i) \in J^r(N,\mathbb{R})_{q_i} \mid h_i : (N,q_i) \to (\mathbb{R},0)\})$

Preliminaries

- $q = (q_1, \ldots, q_s) \in N^{(s)}$
- $g : N \to P$
- $q' = (g(q_1), \ldots, g(q_s))$

- TP : tangent bundle of P
- $g^*TP = \bigcup_{x \in N} T_{g(x)}P$
- $J^r(g^*TP)_{q_i} = \{j^r\xi(q_i) \in J^r(N, g^*TP) \mid \xi : (N, q_i) \to g^*TP, \pi_g \circ \xi = id_{(N,q_i)}\}$,
  where
  - $\pi_g : g^*TP \to N$ $(\pi_g(v_{g(x)}) = x)$
  - $id_{(N,q_i)} : (N, q_i) \to (N, q_i)$: identity
- $J^r(g^*TP)_q = \bigoplus_{i=1}^{s} J^r(g^*TP)_{q_i}$
- $\mathfrak{m}_q J^r(g^*TP)_q$: set consisting of finite sums of the product of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$

Preliminaries

Then, the following canonical identification of $\mathbb{R}$ vector spaces holds.

$$T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_q J^r(g^*TP)_q, \qquad (*)$$

where $q \in N^{(s)}$, $z = {}_sj^r g(q)$ and $q' = (g(q_1), \dots, g(q_s))$.

- W : submanifold of ${}_sJ^r(N,P)$
- Choose $q = (q_1, \dots, q_s) \in N^{(s)}$ and $g : N \to P$ satisfying ${}_sj^r g(q) \in W$
- Set $z = {}_sj^r g(q)$ and $q' = (g(q_1), \dots, g(q_s))$.
- $W_{q,q'}$: the fiber of W over (q, q').

Then, under the identification $(*)$, $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. We denote this vector subspace by $E(g, q, W)$.

Preliminaries

Definition 1 (Mather)

The submanifold W is called *modular* if conditions (α) and (β) below are satisfied:

(α) W is an invariant submanifold of ${}_sJ^r(N,P)$, and lies over P^π for some partition π of $\{1,\ldots,s\}$.

(β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ such that ${}_sj^r g(q) \in W$, the subspace $E(g,q,W)$ is a $J^r(N)_q$-submodule.

For example, contact classes ($\mathcal{K}^r$-orbits) defined by Mather are modular submanifold.

The main theorem of "Generic projections"

The following is the main theorem of "Generic projections".

Theorem 2 (Mather)

- N : *manifold of dimension* n
- $f : N \to \mathbb{R}^m$: *embedding*
- W : *modular submanifold of* ${}_sJ^r(N, \mathbb{R}^\ell)$
- $m > \ell$

$\Rightarrow {}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *Lebesgue measure zero set*
s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$\pi \circ f : N \to \mathbb{R}^\ell$ *is transverse with respect to* W.

An application of the main theorem of “Generic projections”

As one of the applications of the main theorem of “Generic projections”, the following is well known.

Proposition 1 (Mather)

- N : *compact manifold of dimension* n
- $f : N \to \mathbb{R}^m$: *embedding*
- $m > \ell$
- (n, ℓ) : *nice dimensions*

$\Rightarrow$ ${}^{\exists}\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *Lebesgue measure zero set*
s. t. ${}^{\forall}\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$\pi \circ f : N \to \mathbb{R}^\ell$ *is stable.*

The main theorem of my talk

The main theorem of my talk is the following.

Theorem 3 (I)

- N : *manifold of dimension* n
- $f : N \to U$: *embedding* ($U \subset \mathbb{R}^m$: *open*)
- $F : U \to \mathbb{R}^\ell$
- W : *modular submanifold of* ${}_sJ^r(N, \mathbb{R}^\ell)$

$\Rightarrow {}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *Lebesgue measure zero set*
s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$F_\pi \circ f : N \to \mathbb{R}^\ell$ *is transverse with respect to* W.

Remark that in the case of $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$,
Theorem 3 is the main theorem of "Generic projections".

An application of the main theorem of my talk

Proposition 2 (I)

- N : *compact manifold of dimension* n
- $f : N \to U$: *embedding* $(U \subset \mathbb{R}^m$: *open*$)$
- $F : U \to \mathbb{R}^\ell$
- (n, ℓ) : *nice dimensions*

$\Rightarrow {}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: *Lebesgue measure zero set*
s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$F_\pi \circ f : N \to \mathbb{R}^\ell$ *is stable.*

Remark that in the case of $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, Proposition 2 is the previous Mather's proposition (Proposition 1).

Sketch of the proof of the main theorem

In order to give the sketch of the proof of the main theorem of my talk, we recall the theorem as follows.

Theorem 3 (I)

- N : manifold of dimension n
- $f : N \to U$: embedding ($U \subset \mathbb{R}^m$: open)
- $F : U \to \mathbb{R}^\ell$
- W : modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$

$\Rightarrow$ ${}^\exists\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$: Lebesgue measure zero set
s. t. ${}^\forall\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$,
$F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.

Sketch of the proof of the main theorem

(Proof)

- $(\alpha_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$: representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$
- Set $F_\alpha = F_\pi$, and we have

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j}x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j}x_j \right),$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$,
$F = (F_1, \ldots, F_\ell)$ and $x = (x_1, \ldots, x_m)$.

Sketch of the proof of the main theorem

- For the given embedding $f : N \to U$, a mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is as follows:

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right),$$

where $f = (f_1, \ldots, f_m)$.

Sketch of the proof of the main theorem

By the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove the main theorem, it is sufficient to show the following.

Our aim

$^\exists \Sigma \subset (\mathbb{R}^m)^\ell$: Lebesgue measure zero set
s. t. $^\forall \alpha \in (\mathbb{R}^m)^\ell - \Sigma$,
${}_sj^r(F_\alpha \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse
to the given modular submanifold W.

Sketch of the proof of the main theorem

Let $H_\Lambda : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the linear isomorphism defined by

$$H_\Lambda(X_1, \ldots, X_\ell) = (X_1, \ldots, X_\ell)\Lambda,$$

where $\Lambda = (\lambda_{ij})_{1\leq i\leq \ell, 1\leq j\leq \ell}$ is an $\ell \times \ell$ regular matrix.
The composition of H_Λ and $F_\alpha \circ f$ is as follows (in the next page):

Sketch of the proof of the main theorem

$$
\begin{aligned}
& H_\Lambda \circ F_\alpha \circ f \\
= & \left(\sum_{k=1}^{\ell} \left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j \right) \lambda_{k1}, \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f + \sum_{j=1}^{m} \alpha_{kj} f_j \right) \lambda_{k\ell} \right) \\
= & \left(\sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k1} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k1} \alpha_{kj} \right) f_j, \right. \\
& \left. \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k\ell} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k\ell} \alpha_{kj} \right) f_j \right).
\end{aligned}
$$

Sketch of the proof of the main theorem

Set $GL(\ell) = \{B \mid B : \ell \times \ell \text{ matrix}, \det B \neq 0\}$.
Let $\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell$ be the mapping as follows:

$$\varphi(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{\ell m})$$
$$= \Big(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k1}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1},$$
$$\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k2}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{km}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km}\Big).$$

In fact, φ is a C^∞ diffeomorphism.

Sketch of the proof of the main theorem

- Structure of $\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell$

$$
\begin{aligned}
& \varphi(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{\ell m}) \\
= & \Bigg(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{\ell\ell}, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k1}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1}, \\
& \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k2}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k2}, \ldots, \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km}, \sum_{k=1}^{\ell} \lambda_{k2}\alpha_{km}, \ldots, \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km} \Bigg).
\end{aligned}
$$

$$
\begin{aligned}
& H_\Lambda \circ F_\alpha \circ f \\
= & \left(\sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k1} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k1}\alpha_{kj} \right) f_j, \ldots, \sum_{k=1}^{\ell} \left(F_k \circ f \right) \lambda_{k\ell} + \sum_{j=1}^{m} \left(\sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{kj} \right) f_j \right).
\end{aligned}
$$

Sketch of the proof of the main theorem

Next, let $\widetilde{f} : U \to \mathbb{R}^{m+\ell}$ be the mapping as follows:

$$\widetilde{f}(x_1, \ldots, x_m) = (F_1(x_1, \ldots, x_m), \ldots, F_\ell(x_1, \ldots, x_m), x_1, \ldots, x_m).$$

$\Rightarrow \widetilde{f}$: embedding.

Since $f : N \to U$ is an embedding, $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$ is also an embedding:

$$\widetilde{f} \circ f = (F_1 \circ f, \ldots, F_\ell \circ f, f_1, \ldots, f_m).$$

Sketch of the proof of the main theorem

- $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$: embedding
- W : modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$

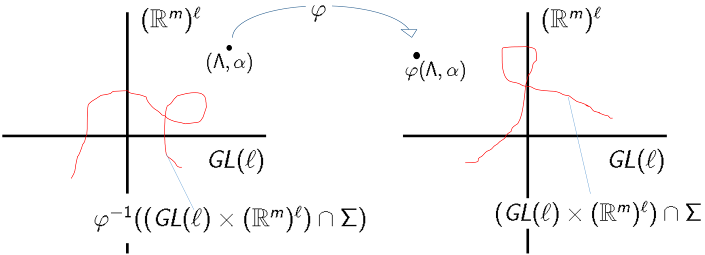
By applying the main theorem of "Generic projections", it follows that
$^\exists \Sigma \subset \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell)$: Lebesgue measure zero set
s. t. $^\forall \Pi \in \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell) - \Sigma$,
${}_sj^r(\Pi \circ \widetilde{f} \circ f) : N^{(s)} \to {}_sJ^r(N, \mathbb{R}^\ell)$ is transverse to W.

Sketch of the proof of the main theorem

By the natural identification $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^{\ell}) = \mathbb{R}^{\ell(m+\ell)}$,
we identify the target space $GL(\ell) \times (\mathbb{R}^m)^\ell$ of the mapping
$\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell$
with an open submanifold of $\mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^{\ell})$.

Sketch of the proof of the main theorem

$$\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell (\subset \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell))$$



- $(GL(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma$: Lebesgue measure zero set of $GL(\ell) \times (\mathbb{R}^m)^\ell$
- $\varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$: Lebesgue measure zero set of $GL(\ell) \times (\mathbb{R}^m)^\ell$

Sketch of the proof of the main theorem

For any $(\Lambda, \alpha) \in GL(\ell) \times (\mathbb{R}^m)^\ell$, let $\Pi_{(\Lambda,\alpha)} : \mathbb{R}^{m+\ell} \to \mathbb{R}^\ell$ be the linear mapping defined by $\varphi(\Lambda, \alpha)$ as follows:

$$
\begin{aligned}
&\Pi_{(\Lambda,\alpha)}(X_1, \ldots, X_{m+\ell}) \\
= \quad & (X_1, \ldots, X_{m+\ell}) \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1\ell} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell 1} & \cdots & \lambda_{\ell\ell} \\ \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{k1} & \cdots & \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{k1} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\ell} \lambda_{k1}\alpha_{km} & \cdots & \sum_{k=1}^{\ell} \lambda_{k\ell}\alpha_{km} \end{pmatrix} .
\end{aligned}
$$

Sketch of the proof of the main theorem

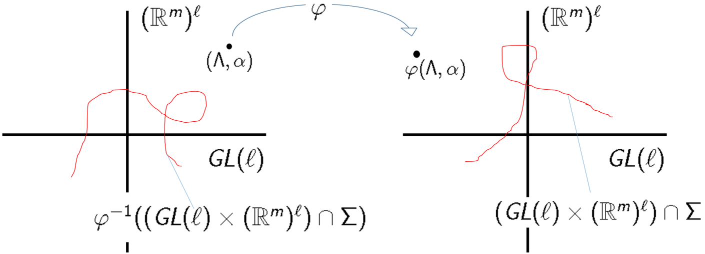
Then, we have the following:

$$
\begin{aligned}
& \Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f \\
= \quad & \left(\sum_{k=1}^{\ell} \Big(F_k \circ f \Big) \lambda_{k1} + \sum_{j=1}^{m} \Big(\sum_{k=1}^{\ell} \lambda_{k1} \alpha_{kj} \Big) f_j, \ldots, \right. \\
& \left. \sum_{k=1}^{\ell} \Big(F_k \circ f \Big) \lambda_{k\ell} + \sum_{j=1}^{m} \Big(\sum_{k=1}^{\ell} \lambda_{k\ell} \alpha_{kj} \Big) f_j \right) \\
= \quad & H_{\Lambda} \circ F_{\alpha} \circ f.
\end{aligned}
$$

Sketch of the proof of the main theorem

Therefore, for any $(\Lambda, \alpha) \in GL(\ell) \times (\mathbb{R}^m)^\ell - \varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$, it follows that ${}_sj^r(\Pi_{(\Lambda,\alpha)} \circ \widetilde{f} \circ f)$ $(= {}_sj^r(H_\Lambda \circ F_\alpha \circ f))$ is transverse to W.

$$\varphi : GL(\ell) \times (\mathbb{R}^m)^\ell \to GL(\ell) \times (\mathbb{R}^m)^\ell (\subset \mathcal{L}(\mathbb{R}^{m+\ell}, \mathbb{R}^\ell))$$



Sketch of the proof of the main theorem

Set $\widetilde{\Sigma} = \{\alpha \in (\mathbb{R}^m)^\ell \mid {}_sj^r(F_\alpha \circ f) \text{ is not transverse to } W\}$.
Suppose $\widetilde{\Sigma}$ is not Lebesgue measure zero set of $(\mathbb{R}^m)^\ell$.
$\Rightarrow$

- $GL(\ell) \times \widetilde{\Sigma}$ is not a Lebesgue measure zero set of $GL(\ell) \times (\mathbb{R}^m)^\ell$.
- ${}^\forall(\Lambda, \alpha) \in GL(\ell) \times \widetilde{\Sigma}$, ${}_sj^r(H_\Lambda \circ F_\alpha \circ f)$ is not transverse to W.

This contradicts the assumption $\varphi^{-1}((GL(\ell) \times (\mathbb{R}^m)^\ell) \cap \Sigma)$ is Lebesgue measure zero.

□

Thank you for your attention.